

B polarization of the CMB from Faraday rotation

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We study the effect of Faraday rotation due to a uniform magnetic field on the polarization of the cosmic microwave background (CMB). Scalar fluctuations give rise only to parity-even E-type polarization of the CMB. However in the presence of a magnetic field, a non-vanishing parity-odd B-type polarization component is produced through Faraday rotation. We derive the exact solution for the E and B modes generated by scalar perturbations including the Faraday rotation effect of a uniform magnetic field, and evaluate their cross-correlations with temperature anisotropies. We compute the angular autocorrelation function of the B-modes in the limit that the Faraday rotation is small. We find that uniform primordial magnetic fields of present strength around $B_0 = 10^{-9}$ G rotate E-modes into B-modes with amplitude comparable to those due to the weak gravitational lensing effect at frequencies around $\nu = 30$ GHz. The strength of B-modes produced by Faraday rotation scales as B_0/ν^2 . We evaluate also the depolarizing effect of Faraday rotation upon the cross correlation between temperature anisotropy and E-type polarization.

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I. INTRODUCTION

The detection of a parity-odd component of the CMB polarization, called B-mode, can provide a unique tool to probe sub-dominant sources of CMB perturbations, which are primarily due to scalar energy-density fluctuations. Scalar perturbations do not produce B polarization, which is only excited by either tensor or vector modes [1, 2]. It is thus important to identify all potential sources of B polarization. A significant background of B-modes comes from the effect of gravitational lensing on the CMB by the matter distribution, which transforms E into B polarization [3]. Cosmological gravitational waves, as those predicted by inflationary models of the early universe, would also imprint B-polarization upon the CMB. It has been pointed out that the signature of an inflationary gravitational-wave background can only be detected by B-polarization measurements if the tensor to scalar ratio $r \geq 10^{-4}$, which corresponds to an energy scale of inflation larger than 3×10^{15} GeV [4, 5], because otherwise their effect would be obscured by the weak lensing background. Quite recently, however, a better technique to *clean* polarization maps from the lensing effect has been proposed, which would allow tensor-to-scalar ratios as low as 10^{-6} , or even smaller, to be probed [6, 7]. Other secondary contributions to the B-type polarization arising during the reionization stage, though of much smaller amplitude, have been considered in [8]. B-modes of polarization are also produced by secondary vector and tensor modes generated by the non-linear evolution of primordial scalar perturbations [9]. Primordial magnetic fields can also generate B-mode polarization upon the CMB, due to the vorticity they induce upon the photon-baryon fluid [10, 11, 12].

We will consider here another source of B-type polarization, that is the Faraday rotation in a magnetic field of initial E-type polarization. If a primordial magnetic field is present at the time of the last scattering of the microwave photons, it causes Faraday rotation of the direction of the linear polarization [13], which generates B-modes from initial E-modes of polarization [14]. In Ref. [14] a cross-correlation between the parity-odd B-modes generated by Faraday rotation and the parity-even temperature anisotropies Θ was estimated under simplifying approximations. Θ -B or E-B correlations also exist if there are other parity violating processes, for instance in the case in which the vorticity in the photon-baryon fluid is induced by helical magnetic fields generated by parity-violating electroweak

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interactions [15, 16], or if parity-violating interactions additional to those in the standard electroweak theory exist [17].

Large scale magnetic fields have been observed in galaxies, clusters and superclusters, but their origin is still unknown [18, 19]. They may have formed from dynamo amplification of a small initial seed field [20], or alternatively they can arise from the adiabatic compression during structure formation of a cosmological field of order $10^{-10} - 10^{-9}$ G today [21]. The existence of these very large scale cosmological fields, although is not excluded by present observations, is difficult to be determined observationally.

Here we derive the exact solution for the E and B modes generated by scalar perturbations including the Faraday rotation effect of a uniform primordial magnetic field, and discuss their properties. We evaluate the effect of Faraday rotation upon the cross correlation Θ -E and the induced correlation Θ -B. We then calculate, to first order in the Faraday rotation effect, the angular correlation function of the B-modes induced by Faraday rotation of scalar E-modes of polarization in a standard cosmological model, and compare it with that due to other potential sources of B-modes, namely weak gravitational lensing, gravitational waves, and the effect of Faraday rotation due to magnetic fields in clusters [22, 23].

II. FARADAY ROTATION OF CMB POLARIZATION

Polarization of the CMB arises from Thomson scattering of anisotropic radiation by free electrons. One way of describing linear polarization is by means of the Stokes parameters Q and U [24]. A magnetic field induces a rotation of the polarization plane of a linearly polarized wave as it propagates through an ionized medium. This effect is known as Faraday rotation. If there were a primordial magnetic field at the recombination epoch, or after the Universe was reionized, it would induce Faraday rotation on the CMB photons, mixing U and Q modes of polarization. The evolution equations for their Fourier modes read [13, 25, 26]

$$\begin{aligned}\dot{Q} + ik\mu Q &= -\dot{\tau}Q + \frac{3}{4}\dot{\tau}(1 - \mu^2)S_p + 2\omega_B U \\ \dot{U} + ik\mu U &= -\dot{\tau}U - 2\omega_B Q,\end{aligned}\tag{1}$$

where $\mu = \hat{n} \cdot \hat{k}$ is the cosine of the angle between the photon direction and the Fourier wave-vector, and $\dot{\tau}$ is the differential optical depth ($\dot{\tau} = n\sigma_T a/a_0$ with σ_T the Thomson scattering cross section, n the free electron density, and the derivative is taken with respect to conformal time $d\eta = dt/a(t)$). We define $\omega_B = 3\vec{B} \cdot \hat{n} c^2 \dot{\tau} / 16\pi^2 \nu^2 e$ as the Faraday rotation rate of radiation with frequency ν in a magnetic field \vec{B} (c is the speed of light and e the electron charge). Polarization is sourced by quadrupole anisotropies in the temperature fluctuations Θ . S_p in eq. (1) is the source term, which reads $S_p = (\Theta_2 + Q_2)/5 + Q_0$ if Θ and Q are expanded in terms of Legendre polynomials as $\Theta = \sum_{\ell} (-i)^{\ell} \Theta_{\ell} P_{\ell}(\mu)$ (and a similar expansion for Q) [36].

The equations above have the following formal integral solutions

$$(Q \pm iU)(\vec{k}, \eta_0) = \frac{3}{4}(1 - \mu^2) \int_0^{\eta_0} d\eta e^{-ik\mu(\eta_0 - \eta)} \dot{\tau} e^{-\tau} S_p(\eta) e^{\mp iF\tau \vec{B} \cdot \hat{n}}.\tag{2}$$

Here $\tau(\eta) = \int_{\eta}^{\eta_0} d\eta' \dot{\tau}(\eta')$ is the optical depth between time η and present time η_0 , and the parameter F is defined as

$$F = \frac{3}{8\pi^2} \frac{Bc^2}{\nu^2 e} \approx 0.7 \left(\frac{B}{10^{-9}\text{G}} \right) \left(\frac{10\text{GHz}}{\nu} \right)^2.\tag{3}$$

F gives a measure of the typical Faraday rotation between collisions. Notice that the strength of a primordial magnetic field is expected to evolve in time $\propto a^{-2}$ due to flux conservation. Since $\nu \propto a^{-1}$, then F (expressed here numerically in terms of the present values of B and ν) does not change with time.

III. E AND B MODES

The generation of temperature and polarization anisotropies in the CMB from gravitational perturbations has been studied in detail by several authors [1, 2, 27, 28]. A simple and powerful formalism is the total angular momentum

method [27, 28], which includes the effect of scalar, vector and tensor modes on an equal footing. We are going to extensively use the results and notation of Ref. [27].

The temperature and polarization fluctuations are expanded in normal modes that take into account the dependence on both the angular direction of photon propagation \hat{n} and the spatial position \vec{x} as

$$\begin{aligned}\Theta(\eta, \vec{x}, \hat{n}) &= \int \frac{d^3k}{(2\pi)^3} \sum_{\ell, m} \Theta_{\ell}^{(m)} G_{\ell}^m, \\ (Q \pm iU)(\eta, \vec{x}, \hat{n}) &= \int \frac{d^3k}{(2\pi)^3} \sum_{\ell, m} (E_{\ell}^{(m)} \pm iB_{\ell}^{(m)})_{\pm 2} G_{\ell}^m,\end{aligned}\tag{4}$$

with spin $s = 0$ describing the temperature fluctuation and $s = \pm 2$ describing the polarization tensor and $m = 0, \pm 1, \pm 2$ corresponding to scalar, vector and tensor perturbations, respectively. $E_{\ell}^{(m)}$ and $B_{\ell}^{(m)}$ are the angular moments of the electric and magnetic polarization components and

$${}_s G_{\ell}^m(\vec{x}, \hat{n}) = (-i)^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} {}_s Y_{\ell}^m(\hat{n}) e^{i\vec{k}\cdot\vec{x}},\tag{5}$$

is the basis of the expansion, in terms of the spin-weighted spherical harmonics.

The set of Boltzmann equations for the evolution of $\Theta_{\ell}^{(m)}$, $E_{\ell}^{(m)}$ and $B_{\ell}^{(m)}$, as well as their integral solutions are deduced in Ref. [27]. In the presence of a uniform magnetic field, Faraday rotation modifies the Boltzmann equations satisfied by $E_{\ell}^{(m)}$ and $B_{\ell}^{(m)}$. We quote in the Appendix the modified equations for completeness. We can nevertheless sidestep them inverting equation (4) to derive an integral solution for $E_{\ell}^{(m)} \pm iB_{\ell}^{(m)}$ from the integral solution for $Q \pm iU$ in eq. (2)

$$E_{\ell}^{(m)} \pm iB_{\ell}^{(m)} = (i)^{\ell} \frac{3}{2} \sqrt{\frac{2\ell+1}{4\pi}} \int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau} P^{(0)} \int d\Omega_{\hat{n}} {}_{\pm 2} Y_{\ell}^{m*}(\hat{n}) (1 - \mu^2) e^{-ik\mu(\eta_0 - \eta)} e^{\mp iF\tau \hat{B}\cdot\hat{n}},\tag{6}$$

where the source term S_p has been expressed in terms of $P^{(0)} = (\Theta_2^{(0)} - \sqrt{6}E_2^{(0)})/10 = S_p/2$ [27]. First the exponential with the magnetic field is expanded in terms of spherical Bessel functions and ordinary spherical harmonics as

$$e^{\mp iF\tau \hat{B}\cdot\hat{n}} = \sum_{\ell, m} i^{\mp \ell} 4\pi j_{\ell}(F\tau) Y_{\ell}^m(\hat{n}) Y_{\ell}^{m*}(\hat{B}).\tag{7}$$

Then the product of one ordinary and one spin-weighted spherical harmonic is written as a sum of terms proportional to just one ${}_s Y_{\ell}^m(\hat{n})$ using the Clebsh-Gordan relations for the addition of angular momentum [27]:

$$\begin{aligned}({}_{s_1} Y_{\ell_1}^{m_1}) ({}_{s_2} Y_{\ell_2}^{m_2}) &= \frac{\sqrt{(2\ell_1+1)(2\ell_2+1)}}{4\pi} \sum_{\ell, m, s} \langle \ell_1, \ell_2; m_1, m_2 | \ell_1, \ell_2; \ell, m \rangle \\ &\times \langle \ell_1, \ell_2; -s_1, -s_2 | \ell_1, \ell_2; \ell, -s \rangle \sqrt{\frac{4\pi}{2\ell+1}} ({}_s Y_{\ell}^m).\end{aligned}$$

The term $\exp(-ik\mu(\eta_0 - \eta))$ is also expanded in terms of spherical Bessel functions, with the \hat{e}_3 axis chosen in the direction of \vec{k} (which leaves only $Y_{\ell}^0(\hat{n})$ in the expansion) and the angular integration is performed. The final result is

$$\begin{aligned}E_{\ell}^{(m)} \pm iB_{\ell}^{(m)} &= (-1)^{m+1} \sqrt{24\pi} (2\ell+1) \int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau} P^{(0)} \sum_{\ell' \ell''} (i)^{\ell - \ell'' \mp \ell'} Y_{\ell'}^{m*}(\hat{B}) j_{\ell'}(F\tau) \sqrt{2\ell'+1} \times \\ &\times \langle \ell, \ell'; -m, m | \ell, \ell'; \ell'', 0 \rangle \langle \ell, \ell'; \pm 2, 0 | \ell, \ell'; \ell'', \pm 2 \rangle \epsilon_{\ell''}^{(0)}(k(\eta_0 - \eta)).\end{aligned}\tag{8}$$

where $\epsilon_{\ell}^{(0)}(x) = \sqrt{3(l+2)!/8(l-2)!} j_{\ell}(x)/x^2$.

Using the relation $\langle \ell, \ell'; -2, 0 | \ell, \ell'; \ell'', -2 \rangle = (-1)^{\ell'' - \ell + \ell'} \langle \ell, \ell'; +2, 0 | \ell, \ell'; \ell'', +2 \rangle$ between Clebsh-Gordan coefficients, individual expressions for $E_{\ell}^{(m)}$ and $iB_{\ell}^{(m)}$ immediately follow. They are similar to the right hand side of eq. (8) choosing the upper sign, with an extra factor $(1 + (-1)^{\ell'' - \ell})/2$ in the case of $E_{\ell}^{(m)}$ and $(1 - (-1)^{\ell'' - \ell})/2$ in the case of $iB_{\ell}^{(m)}$ inside the sum.

Notice that in the absence of a magnetic field the standard solution [27] for $E_\ell^{(0)}$ is recovered, and also $B_\ell^{(m)} = 0$ as it should. Instead, in the limit of relatively large magnetic field ($F \gg 1$), both $E_\ell^{(m)}$ and $B_\ell^{(m)}$ tend to zero due to the rapidly oscillatory nature of the spherical Bessel functions. This is a manifestation of the depolarizing effect of differential Faraday rotation [25].

The presence of the magnetic field, which breaks the invariance under rotations, gives rise to a coupling between different m multipoles. This is evident from eq. (8), where a full range of m polarization multipoles is obtained from the initial scalar $m = 0$ modes, as well as from the Boltzmann equations displayed in the Appendix.

IV. FARADAY ROTATION EFFECT UPON CROSS-CORRELATIONS

In order to calculate correlation functions one has to integrate over wavevectors \vec{k} , for which our expressions for $\Theta_\ell^{(m)}$, $E_\ell^{(m)}$ and $B_\ell^{(m)}$, valid in a frame with $\hat{e}_3 \parallel \vec{k}$, need to be rotated to a frame with fixed axis. In the absence of a magnetic field, statistical homogeneity and isotropy makes the angular integrations simple, and the angular power spectrum of the correlation functions can be calculated as

$$C_\ell^{XZ} = \frac{2}{\pi} \int k^2 dk \sum_m \frac{X_\ell^{(m)*}(\eta_0, k) Z_\ell^{(m)}(\eta_0, k)}{(2\ell + 1)^2} \quad (9)$$

with $X, Z = \Theta, E, B$. Statistical homogeneity and isotropy is reflected in the fact that

$$\langle X_{\ell m} Z_{\ell' m'} \rangle = \delta_{mm'} \delta_{\ell\ell'} C_\ell^{XZ}, \quad (10)$$

where $X_{\ell m}$ are the coefficients of the multipole expansions in the fixed basis:

$$\begin{aligned} \Theta(\hat{n}) &= \sum_{\ell, m} \Theta_{\ell m} Y_{\ell m} \\ (Q \pm iU)(\hat{n}) &= \sum_{\ell, m} (E_{\ell m} \pm iB_{\ell m})_{\pm 2} Y_\ell^m. \end{aligned} \quad (11)$$

A uniform magnetic field breaks invariance under rotations and defines a preferred direction in space. We can take the direction of the magnetic field as the \hat{e}_3 direction of the fixed basis. The multipole coefficients $E_{\ell m} \pm iB_{\ell m}$ in this basis can be calculated through a rotation of those given by eq. (8), as

$$E_{\ell m} \pm iB_{\ell m} = i^\ell \frac{4\pi}{2\ell + 1} \int \frac{d^3 \vec{k}}{2\pi^3} e^{i\vec{k} \cdot \vec{x}} \sum_{m'} (E_\ell^{(m')} \pm iB_\ell^{(m')})_{-m'} Y_\ell^{m*}(\hat{k}). \quad (12)$$

Here $E_\ell^{(m')} \pm iB_\ell^{(m')}$ are given by eq. (8) with \hat{B} replaced by \hat{k} in the argument of $Y_\ell^{m'}$, since now the angle between \vec{B} and \vec{k} is measured from the fixed \hat{e}_3 axis along \vec{B} . The factor summed over m' in eq. (12) corresponds to the rotation of the polarization multipoles corresponding to each wavevector \vec{k} mode to the fixed axis system, multiplied by $\sqrt{(2\ell + 1)/4\pi}$. The other factors arise from the definition of the basis $_{\pm 2}G_\ell^m$ used in eq. (4). The product of two spherical harmonics appearing after replacing eq. (8) in this expression can be expanded with the relations for addition of angular momentum, with the help of the relation $(-1)^m {}_m Y_\ell^0(\theta, \phi) = Y_\ell^m(\theta, 0)$, and the sum over m' can be done using the orthogonality properties of the Clebsch-Gordan coefficients. The result can finally be expressed in terms of the analogous multipole coefficients in the absence of a magnetic field as [37]

$$E_{\ell m} \pm iB_{\ell m} = \sum_{\ell', \ell''} \left[\frac{(-1)^{\ell'' - \ell} + 1}{2} \pm \frac{(-1)^{\ell'' - \ell} - 1}{2} \right] R(\ell m, \ell', \ell'' m) \tilde{E}_{\ell'' m}^{(\ell')}. \quad (13)$$

Here $\tilde{E}_{\ell m}^{(\ell')}$ coincides with the expression for the multipole coefficients due to scalar fluctuations in the absence of a magnetic field, except for an extra term $(2\ell' + 1)j_{\ell'}(F\tau)$ inside the time-integral that defines it. Explicitly:

$$\tilde{E}_{\ell m}^{(\ell')} = -i^\ell \frac{4\pi}{2\ell + 1} \int \frac{d^3 \vec{k}}{2\pi^3} e^{i\vec{k} \cdot \vec{x}} Y_\ell^{m*}(\hat{k}) \sqrt{6(2\ell + 1)} \int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau} (2\ell' + 1) j_{\ell'}(F\tau) P^{(0)}(\eta) \epsilon_\ell^{(0)}(k(\eta_0 - \eta)). \quad (14)$$

The coefficients $R(\ell m, \ell', \ell'' m'')$ are given, in terms of Clebsch-Gordan coefficients, by

$$R(\ell m, \ell', \ell'' m'') = i^{\ell'} \frac{\sqrt{2\ell'' + 1}}{\sqrt{2\ell + 1}} \langle \ell', \ell''; m - m'', m'' | \ell' \ell''; \ell, m \rangle \langle \ell', \ell''; 0, 2 | \ell' \ell''; \ell, 2 \rangle. \quad (15)$$

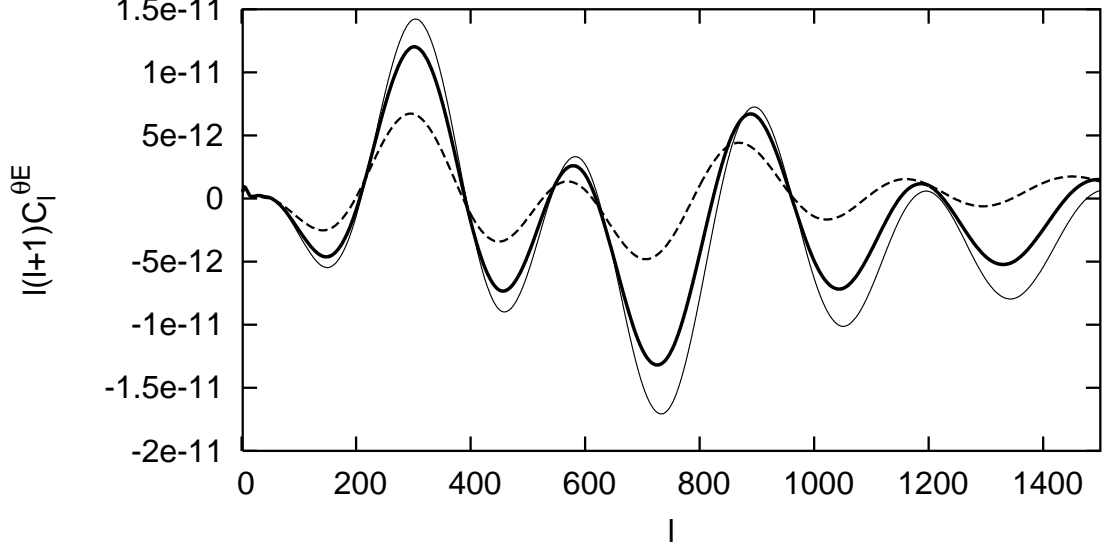


FIG. 1: Temperature-polarization (E-mode) cross-correlation in a Λ CDM background cosmology in the case of no magnetic field (thin solid line) and in the case of Faraday rotation in a uniform magnetic field with $F = 1$ (thick solid line) and $F = 3$ (dashed line). F is defined in eq. (3). For large F the depolarizing effect of differential Faraday rotation is quite noticeable.

Correlation functions including the Faraday rotation effect can now be immediately calculated in terms of their values in the absence of the magnetic field, as given by eq. (10). The multipoles of the cross-correlation between temperature anisotropies Θ and E and B modes can be written as

$$\langle \Theta_{\ell_1 m_1}^* (E_{\ell_2 m_2} \pm i B_{\ell_2 m_2}) \rangle = \left[\frac{(-1)^{\ell_2 - \ell_1} + 1}{2} \pm \frac{(-1)^{\ell_2 - \ell_1} - 1}{2} \right] \delta_{m_1 m_2} \sum_{\ell'} R(\ell_2 m_2, \ell', \ell_1 m_1) C_{\ell_1}^{\Theta \tilde{E}^{(\ell')}}. \quad (16)$$

This expression is strictly valid in a frame with $\hat{e}_3 \parallel \vec{B}$, and the fact that only objects with the same m are correlated reflects the cylindrical symmetry of the problem around this axis. The cross-correlation in any other system can be obtained by noticing that the multipoles in a new system are related to the ones in the $\hat{e}_3 \parallel \vec{B}$ system by $X'_{\ell m} = \sqrt{4\pi/2\ell+1} \sum_{m'} X_{\ell m' - m'} Y_{\ell}^{\prime m*}(\hat{B})$. In the general system the cross correlation results

$$\langle \Theta_{\ell_1 m_1}^* (E'_{\ell_2 m_2} \pm i B'_{\ell_2 m_2}) \rangle = \left[\frac{(-1)^{\ell_2 - \ell_1} + 1}{2} \pm \frac{(-1)^{\ell_2 - \ell_1} - 1}{2} \right] \sum_{\ell'} R(\ell_2 m_2, \ell', \ell_1 m_1) \sqrt{\frac{4\pi}{2\ell' + 1}} Y_{\ell'}^{\prime m_2 - m_1*}(\hat{B}) C_{\ell_1}^{\Theta \tilde{E}^{(\ell')}}. \quad (17)$$

Thus, in a general system the correlations between different m values do not vanish. This fact can be used to identify the presence of a symmetry axis, that in this case corresponds to the direction of the magnetic field.

It is clear from the expressions in eqs. (16) and (17) that Faraday rotation in a uniform magnetic field generates non-diagonal ($\ell_1 \neq \ell_2$) correlations Θ -B out from initially diagonal correlations Θ -E. The difference between ℓ_1 and ℓ_2 must be odd, and contributions to the case $|\ell_2 - \ell_1| = 2k + 1$ are increasingly suppressed for larger k . They are in fact of order larger or equal than $2k + 1$ in the parameter F . This is because the coefficient R vanishes if $|\ell_2 - \ell_1| > \ell'$ and since $j_{\ell'}(F) \propto F^{\ell'}$ for small F . Likewise, Faraday rotation induces E-B correlations with $\ell_2 - \ell_1$ odd. It also distorts the diagonal Θ -E and E-E correlations, and induces non-diagonal terms with $|\ell_2 - \ell_1| = 2k$, proportional at least to F^{2k} for small F .

In the system with $\hat{e}_3 \parallel \vec{B}$, we can actually calculate to all orders the effect of Faraday rotation upon the diagonal part of the correlation Θ -E. Indeed, since $\sum_m R(\ell m, \ell', \ell m) = (2\ell + 1)\delta_{\ell' 0}$ we find that

$$C_{\ell}^{\Theta E} \equiv \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \langle \Theta_{\ell m}^* E_{\ell m} \rangle = C_{\ell}^{\Theta \tilde{E}^{(0)}}. \quad (18)$$

In other words, the diagonal terms in the cross-correlation Θ -E can be calculated, to all orders, exactly as in the case of no magnetic field with the addition of a term $j_0(F\tau)$ in the integrand along the line of sight for $E_{\ell m}$. Notice however that this equivalence is not valid for individual contributions to C_{ℓ} with different values of m , since they also

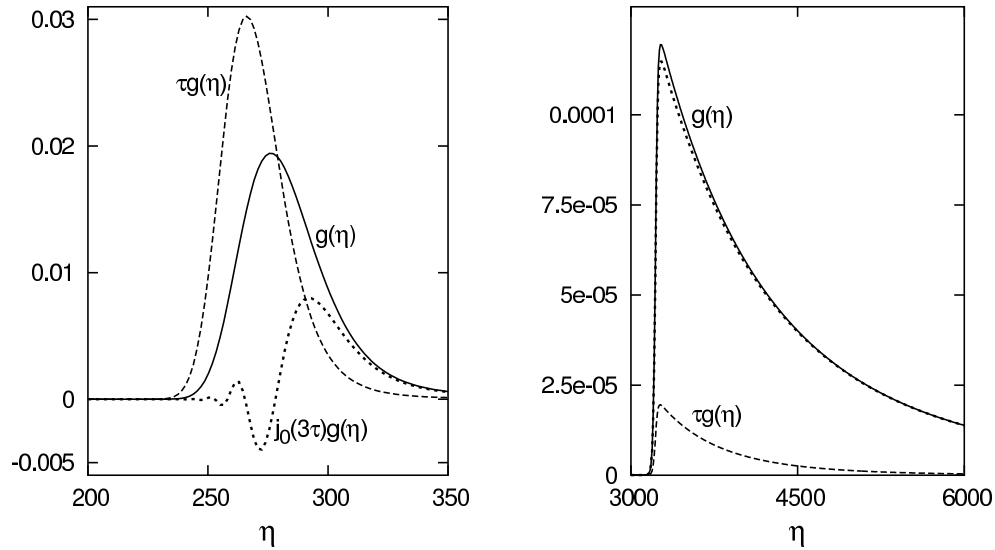


FIG. 2: Visibility function $g(\eta)$ (solid line) as a function of conformal time in the background Λ CDM cosmology, calculated with CMBFAST, and multiplied by $j_0(F\tau)$ with $F = 3$ (dotted line), and multiplied by τ (dashed line). The left panel displays times around hydrogen recombination and photon decoupling. The right panel displays times after reionization, which was assumed to take place at an optical depth $\tau_{\text{reion}} = 0.17$.

get additional contributions from higher $\tilde{E}_{\ell m}^{(\ell')}$ which, however, add to zero in the sum over m . We can readily evaluate expression (18) numerically using CMBFAST [29], account taken of the extra term $j_0(F\tau)$. As background cosmology a spatially-flat Λ -CDM model was assumed, with 70% of the energy density in the form of a cosmological constant, 25.6% in cold dark matter, 4.4% in baryons, a Hubble constant $H_0 = 71$ km/sec Mpc, adiabatic scale invariant scalar fluctuations, and an optical depth since reionization $\tau_{\text{reion}} = 0.17$, as suggested by WMAP measurements of temperature-polarization correlations [31]. The result is plotted in Figure 1, in the standard case of no Faraday rotation (thin solid line), and in the case of Faraday rotation factors $F = 1$ (thick solid line) and $F = 3$ (dashed line).

Notice that there is no linear effect of Faraday rotation upon $C_\ell^{\Theta E}$ for small F , the lowest order correction being quadratic and with negative sign. For large F the oscillatory nature of $j_0(F\tau)$ leads to a significant reduction in the amplitude of the cross-correlation, reflecting the depolarizing effect of large differential Faraday rotation [25]. A shift towards larger angular scales (smaller ℓ) in the positions of the peaks is also noticeable in Fig. 1. This can be interpreted as the consequence of the fact that the integrand in eq. (14) for $\tilde{E}_{\ell m}^{(0)}$ contains the factor $g(\eta)j_0(F\tau)$, where $g(\eta) = \dot{\tau} \exp(-\tau)$ is the visibility function (the probability that a photon last-scattered within $d\eta$ of η). The visibility function around the time of hydrogen recombination can be well approximated by a Gaussian centered at the time of photon decoupling. The polarization produced by Thomson scattering at recombination can be well approximated as proportional to the time derivative of temperature inhomogeneities around decoupling and to the width of the visibility function. The additional factor $j_0(F\tau)$ in the integrand of eq. (14) not only reduces the value of the integral, but also shifts to later times the location of the largest contribution. We display this effect in Fig. 2, where we plot the visibility function $g(\tau)$ (solid line) and the factor $g(\eta)j_0(F\tau)$ for $F = 3$ (dotted line), both around the time of photon decoupling (left panel) and after reionization (right panel). The shift to later times in the location of the peak in $g(\eta)j_0(F\tau)$ in the left panel translates into a shift towards larger angular scales in the position of the peaks in the correlation function.

Faraday rotation has a relatively smaller effect upon E modes produced after reionization because τ is not very large (smaller than 0.17 in our numerical examples), and thus $j_0(F\tau)$ is closer to unity unless F is larger.

The strong damping of ΘE correlations that occurs when F is of order unity or larger also affects ΘB , EB , EE and BB correlations. It is a consequence that differential Faraday rotation across the last scattering surface (or after

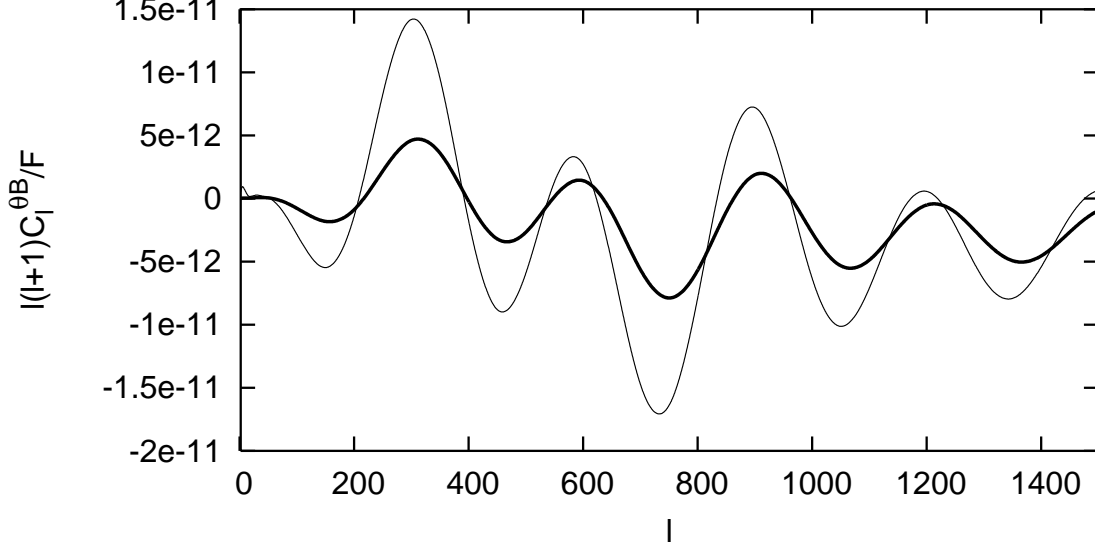


FIG. 3: Temperature-polarization (B-mode) cross-correlation (thick solid line) in a Λ CDM background cosmology due to Faraday rotation of E-modes, scaled by the factor F defined in eq. (3), assumed to be smaller than unity. See the text for the definition of $C_l^{\Theta B}$, which is actually a non-diagonal correlation. The correlation Θ -E in the absence of magnetic field is also displayed for reference (thin solid line).

reionization if F were large enough) has a net depolarizing effect.

Corrections to $C_\ell^{\Theta E}$ smaller than those considered here exist due to the fact that Faraday rotation affects the quadrupole term in the temperature anisotropy source [25], an effect that we have neglected.

It is clear from Fig. 1 that Faraday rotation factors of order unity would imprint a measurable signature upon the temperature-polarization cross-correlation, which is moreover frequency-dependent. Primordial magnetic fields with strength 10^{-8} G (scaled to present values) can easily be ruled out with measurements at frequencies of order 30 GHz.

V. CORRELATION FUNCTIONS FOR SMALL FARADAY ROTATION

In the (likely to be realistic) case of small Faraday rotation ($F \ll 1$), it suffices to take $\ell' = 1$ in eq. (16) or (17). To this order, the cross-correlation Θ -B in the system with $\hat{e}_3 \parallel \vec{B}$ reads

$$\langle \Theta_{\ell_1 m}^* B_{\ell_2 m} \rangle = C_{\ell_1}^{\Theta \tilde{E}^{(1)}} \left[\delta_{\ell_2, \ell_1+1} \frac{\sqrt{(\ell_1+1)^2 - m^2} \sqrt{(\ell_1+1)^2 - 4}}{(\ell_1+1)\sqrt{2\ell_1+1}\sqrt{2\ell_1+3}} + \delta_{\ell_2, \ell_1-1} \frac{\sqrt{\ell_1^2 - m^2} \sqrt{\ell_1^2 - 4}}{\ell_1\sqrt{2\ell_1-1}\sqrt{2\ell_1+1}} \right]. \quad (19)$$

In a general frame there are also correlations among multipoles with values of m differing by ± 1 . Notice that a similar expression holds for the cross correlation E-B, with $C_{\ell_1}^{\tilde{E}^{(0)} \tilde{E}^{(1)}}$ in place of $C_{\ell_1}^{\Theta \tilde{E}^{(1)}}$.

In Fig. 3 we plot, as a measure of the strength of the cross-correlations Θ -B, the quantity

$$C_\ell^{\Theta B} \equiv C_\ell^{\Theta \tilde{E}^{(1)}} \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \frac{\sqrt{\ell^2 - m^2} \sqrt{\ell^2 - 4}}{\ell \sqrt{2\ell-1} \sqrt{2\ell+1}}, \quad (20)$$

which represents an average over m of the amplitude of the non-diagonal correlations. We plot it scaled down by the factor F , assumed to be small. Notice that $C_\ell^{\Theta \tilde{E}^{(1)}}$ coincides with the power spectrum for Θ -E correlations produced by scalar fluctuations in the absence of a magnetic field except for an extra factor $3j_1(F\tau) \approx F\tau$ in the time integral that leads to E. This can be interpreted as the fact that the B-modes are the result of the E-modes produced by Thomson scattering up to a time η , further Faraday rotated by a factor $F\tau$ since, and integrated over η . Fig. 2 displays the factors $g(\eta)$ and $\tau g(\eta)$ in the background cosmology chosen for illustration, both around photon decoupling (left panel) and after reionization (right panel). $\tau g(\eta)$ has its peak shifted to earlier times around decoupling compared with the visibility function. This effect shifts the peaks in the correlation functions to smaller angular scales (larger

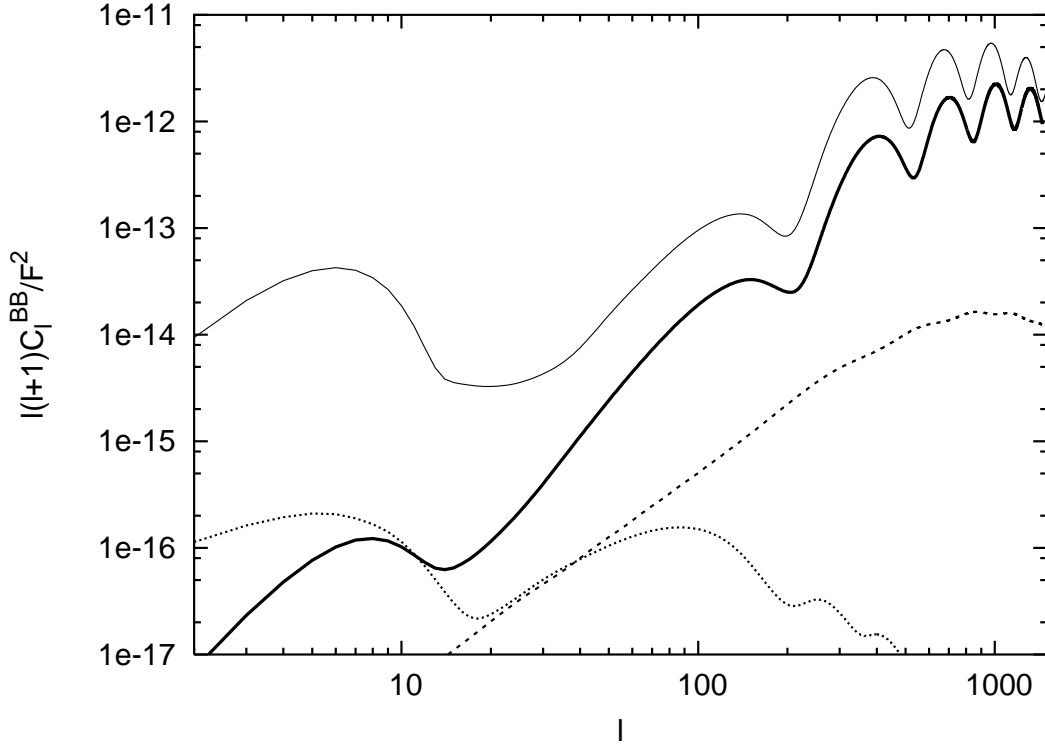


FIG. 4: Angular autocorrelation multipoles of B-modes of CMB polarization generated by Faraday rotation of E-modes generated by scalar perturbations in a Λ CDM cosmology, scaling away the factor F^2 (thick solid line). The autocorrelation of E-modes from which they originate is also displayed (thin solid line) for comparison purposes. Also shown are the autocorrelation of B-modes due to weak gravitational lensing (dashed line) and to gravitational waves in a model with tensor to scalar ratio $r = 10^{-2}$ (dotted line).

values of ℓ). Faraday rotation effects after reionization are not only suppressed by a factor F , but also by the relatively smaller values of τ , as displayed in the right panel.

The diagonal contribution to the angular autocorrelation of B-modes is given, to lowest order in F and averaged over m , by

$$C_\ell^{BB} \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle B_{\ell m}^* B_{\ell m} \rangle = \frac{1}{3} \frac{(\ell+1)^2 - 4}{(2\ell+3)(\ell+1)} C_{\ell+1}^{\tilde{E}^{(1)} \tilde{E}^{(1)}} + \frac{1}{3} \frac{\ell^2 - 4}{(2\ell+1)\ell} C_{\ell-1}^{\tilde{E}^{(1)} \tilde{E}^{(1)}}. \quad (21)$$

We calculate it numerically using CMBFAST [29], accounting for the extra factors $F\tau$ in the integrals along the line of sight for $\tilde{E}^{(1)}$. The result is displayed in Figure 4, for the same background Λ -CDM cosmological model as in the previous section. The quantity plotted (thick solid line) is $\ell(\ell+1)C_\ell^{BB}/F^2$, with F given by eq. (3), so that the result should be scaled down by the appropriate value of F^2 , which we have assumed to be small. We also plot, for comparison purposes, the B-mode correlation due to weak gravitational lensing (dashed line), and that due to gravitational waves in a model with tensor to scalar ratio $r = 10^{-2}$ (following the convention in e.g. [32] for its definition). Also plotted in Fig. 4, to help estimate the strength of the effect, is $\ell(\ell+1)C_\ell^{EE}$ due to the scalar fluctuations in the absence of Faraday rotation (thin solid line).

The angular correlation multipoles of B-modes are roughly a factor $F^2/3$ smaller than those of the E-modes from which they originate by Faraday rotation, at relatively large ℓ . Notice also the shift in the peaks positions towards larger ℓ . At small ℓ the effect is relatively suppressed because the B-modes produced by Faraday rotation are smaller than the E-modes generated after reionization by an extra factor somewhat smaller than τ_{reion}^2 (the optical depth to reionization).

It is clear from Fig. 4 that Faraday rotation factors of order $F = 0.1$, which at $\nu = 30$ GHz correspond to a primordial magnetic field of present strength $B_0 = 10^{-9}$ G, induce B-modes with amplitude comparable to those generated by weak gravitational lensing. B-modes due to Faraday rotation in the magnetic fields of clusters of galaxies could have

a comparable strength (but different angular spectrum) around $\ell = 1000$ only if $B_{\text{clusters}} \approx 2 \mu\text{G}$, uniform across the clusters [22, 23].

VI. CONCLUSIONS

The detection of polarization in the CMB [30] and its correlation to the temperature anisotropies [31] has become a very promising tool to test the early universe. Much observational effort is taking place to measure Θ , E and B correlation functions. The cross-correlation function ΘE already gives information on the reionization history of the universe. The B-mode polarization is being looked upon specially as a way to probe a primordial gravitational wave background which would represent a clear signature of a period of inflation in the early Universe. Precision data on the CMB polarization is expected from the Planck Mission [33] and future dedicated missions, such as NASA's *Beyond Einstein* Inflation Probe [34] or ground-based experiments, like *BICEP* [35] and *PolarBear*.

There are other processes beside a gravitational wave background that can originate B-mode polarization, more prominently the gravitational lensing conversion of a fraction of the dominant E-type into B-type polarization, which can however be significantly cleaned from the data allowing to probe inflationary models with tensor-to-scalar ratios $r \leq 10^{-6}$ [7].

We have studied here in detail the effect of the Faraday rotation produced by a large scale uniform primordial magnetic field upon the polarization of the CMB for the case of scalar density perturbations. The exact solution for the E and B modes of polarization in the presence of a uniform magnetic field is given in eq. (13). Faraday rotation of the direction of the polarization generates a B-polarization component from the initial E-modes, which correlates with temperature anisotropies and E-polarization [14]. Θ -B correlations are non-diagonal, and their strength is of order F times the original Θ -E correlations, with the parameter F defined by eq. (3), as long as F is smaller than unity. The amplitude of the B-polarization angular autocorrelation function generated by Faraday rotation is comparable to the (uncleaned) weak lensing signal if $F \sim 0.1$, which corresponds to a magnetic field $B \simeq 10^{-9}\text{G}$ at a frequency $\nu \simeq 30\text{GHz}$. Let us notice however, that the lensing signal, as well as that generated by gravitational waves, are independent of the frequency, while F scales as ν^{-2} , thus multi-frequency measurements should easily distinguish the Faraday rotation contribution.

Small Faraday rotation factors have a relatively small effect upon ΘE and EE correlations, since they induce distortions roughly F^2 smaller than their values in the absence of a magnetic field. If F were of order unity or larger, all correlations ΘE , ΘB , EE , EB and BB are significantly damped, due to the additional depolarizing effect of differential Faraday rotation across the last scattering surface. The good agreement between WMAP measurements [31] of ΘE correlations and its predicted value in a standard cosmology rules out values of F of order unity, which implies primordial magnetic fields of present strength $B_0 \approx 10^{-8}\text{G}$ at frequencies $\nu = 30\text{GHz}$. Future precision measurements should allow to probe smaller values of primordial magnetic fields through Faraday rotation effects.

APPENDIX: BOLTZMANN EQUATIONS FOR E AND B

We write down here the Boltzmann equations for the E and B-modes of polarization including the Faraday rotation effect of a constant magnetic field. We adopt the formalism of Ref. [27] for the case without a magnetic field. The additional terms, that mix E and B modes, can be obtained by derivation of the defining equation (4), using eq. (1), and the Clebsh-Gordan relations for products of spherical harmonics. It turns out that

$$\begin{aligned}
\dot{E}_\ell^{(m)} = & k \left[\frac{2\kappa_\ell^m}{(2\ell-1)} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)} B_\ell^{(m)} - \frac{2\kappa_{\ell+1}^m}{(2\ell+3)} E_{\ell+1}^{(m)} \right] - \dot{\tau} [E_\ell^{(m)} + \sqrt{6} P^{(m)} \delta_{\ell,2}] \\
& + iF\dot{\tau} \left[\cos\theta_B \left(-\frac{2\kappa_{\ell+1}^m}{2\ell+3} B_{\ell+1}^{(m)} + 2\frac{m}{\ell(\ell+1)} E_\ell^{(m)} + \frac{2\kappa_\ell^m}{2\ell-1} B_{\ell-1}^{(m)} \right) \right. \\
& - \frac{1}{\sqrt{2}} \sin\theta_B e^{-i\phi_B} \left(\frac{2\gamma_{\ell+1}^{-m}}{2\ell+3} B_{\ell+1}^{(m-1)} - 2\alpha_\ell^{m-1} E_\ell^{(m-1)} + \frac{2\gamma_\ell^{m-1}}{2\ell-1} B_{\ell-1}^{(m-1)} \right) \\
& \left. + \frac{1}{\sqrt{2}} \sin\theta_B e^{i\phi_B} \left(\frac{2\gamma_{\ell+1}^m}{2\ell+3} B_{\ell+1}^{(m+1)} + 2\alpha_\ell^m E_\ell^{(m+1)} + \frac{2\gamma_\ell^{-(m+1)}}{2\ell-1} B_{\ell-1}^{(m+1)} \right) \right] \quad (\text{A.1})
\end{aligned}$$

and

$$\begin{aligned}
\dot{B}_\ell^{(m)} = & k \left[\frac{2\kappa_\ell^m}{(2\ell-1)} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)} E_\ell^{(m)} - \frac{2\kappa_{\ell+1}^m}{(2\ell+3)} B_{\ell+1}^{(m)} \right] - \dot{\tau} B_\ell^{(m)} \\
& + iF\dot{\tau} \left[\cos\theta_B \left(\frac{2\kappa_{\ell+1}^m}{2\ell+3} E_{\ell+1}^{(m)} + 2\frac{m}{\ell(\ell+1)} B_\ell^{(m)} - \frac{2\kappa_\ell^m}{2\ell-1} E_{\ell-1}^{(m)} \right) \right. \\
& - \frac{1}{\sqrt{2}} \sin\theta_B e^{-i\phi_B} \left(-\frac{2\gamma_{\ell+1}^m}{2\ell+3} E_{\ell+1}^{(m-1)} - 2\alpha_\ell^{m-1} B_\ell^{(m-1)} - \frac{2\gamma_\ell^{m-1}}{2\ell-1} E_{\ell-1}^{(m-1)} \right) \\
& \left. + \frac{1}{\sqrt{2}} \sin\theta_B e^{i\phi_B} \left(-\frac{2\gamma_{\ell+1}^m}{2\ell+3} E_{\ell+1}^{(m+1)} + 2\alpha_\ell^m B_\ell^{(m+1)} - \frac{2\gamma_\ell^{-(m+1)}}{2\ell-1} E_{\ell-1}^{(m+1)} \right) \right] \quad (\text{A.2})
\end{aligned}$$

where θ_B and ϕ_B are the polar and azimuthal angles respectively of the direction of the magnetic field in a system with $\hat{e}_3 \parallel \vec{k}$, and we have defined

$$\begin{aligned}
{}_s\kappa_\ell^m &= \frac{\sqrt{(\ell^2 - m^2)(\ell^2 - s^2)}}{\ell} \\
{}_s\gamma_\ell^m &= \frac{\sqrt{(\ell^2 - s^2)(\ell + m)(\ell + m + 1)}}{\sqrt{2}\ell} \\
\alpha_\ell^m &= \frac{\sqrt{(\ell - m)(\ell + m + 1)}}{\sqrt{2}\ell(\ell + 1)}
\end{aligned}$$

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- [37] If the fluctuations were such that B-modes exist in the absence of a magnetic field, then equation (13) is easily generalized with $\tilde{E}_{\ell''m}^{(\ell')} \pm i\tilde{B}_{\ell''m}^{(\ell')}$ in its right hand side.